A review of numerical methods for fractional ordinary differential equations

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Organizacja IX Konferencji Modelowanie Matematyczne w Fizyce i Technice (MMFT 2017) - zadanie finansowane w ramach umowy 829/P-DUN/2017 ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych na działalność upowszechniającą naukę.

Formulation of the problem

In recent years there has been an increase in the number of publications devoted to differential equations of fractional order, which are widely applied in modeling many problems in: physics, control theory, bioengineering and mechanics [1,2,3]. In many cases, obtaining an analytical solution for fractional differential equations is very difficult, or even impossible, then we apply numerical methods. Consider a one-term fractional differential equation including the left-sided Caputo derivative:

$$ {}^{c}D^{\alpha}_{0+}f(t) = \psi(t, f(t)), \qquad \alpha \in (0,1] \tag{1}$$

with initial condition

$$f(0) = f_0. \tag{2}$$

The starting point for the all numerical methods discussed in the paper is transformation of the initial value problem (1-2) into an equivalent integral equation:

$$f(t) = I^{\alpha}_{0+}\psi(t, f(t)) + f(0). \tag{3}$$

We compare numerical results obtained by Euler method [4] and two variants of Adams-Bashforth-Moulton (A-B-M) method [4,5]. In Euler method we apply rectangle rule to calculate integral in formula (3). First variant of (A-B-M) method requires trapezoidal rule to calculate corrector. The second one requires two methods to determine the corrector: Simpson's rule or trapezoidal rule depending on an odd or even number of nodes in the integration interval.

Numerical examples

Consider the following simple version of equation (1):

$$ {}^{c}D^{\alpha}_{0+}f(t) = f(t) \tag{4}$$

supplemented with initial condition

$$f(0) = 1. \tag{5}$$

The exact solution of the initial value problem (4-5) is given by the two parameters Mittag-Leffler function:

$$f(t) = E_{\alpha,1}(t^{\alpha}). \tag{6}$$

Table 1, 3 and 5 shows the average errors of the three tested numerical schemes resulting from the comparison of its exact and numerical solutions. Table 2, 4 and 6 contains the results of experimental estimation of the convergence order (EOC) calculated for Euler, A-B-M-first variant and A-M-B-second variant methods.

Tablica 1: Error generated by the Euler method.

h	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1/10	1.65	3.58e-1	1.18e-1	4.77e-2
1/50	4.45e-1	7.71e-2	2.42e-2	9.9e-3
1/250	8.93e-2	1.5e-2	4.87e-3	2e-3
1/1000	2.14e-2	3.67e-3	1.21e-3	5e-4

Tablica 2: EOC generated by the Euler method.

h	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1/10	0.661	0.834	0.866	0.852
1/50	0.967	1.004	0.988	0.927
1/250	1.033	1.02	1.003	0.994
1/1000	1.034	1.014	1.002	0.998

Tablica 3: Error generated by the A-B-M method-first variant.

h	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1/10	7.94e-1	9.29e-2	1.42e-2	1.6e-3
1/50	1.35e-1	9.57e-3	9.78e-4	6.61e-5
1/250	1.89e-2	8.92e-4	6.18e-5	2.66e-6
1/1000	3.36e-3	1.13e-4	5.76e-6	1.67e-7

Tablica 4: EOC generated by the A-B-M method-first variant.

h	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1/10	0.958	1.159	1.474	1.751
1/50	1.198	1.445	1.662	1.949
1/250	1.242	1.485	1.71	1.99
1/1000	1.249	1.494	1.726	1.997

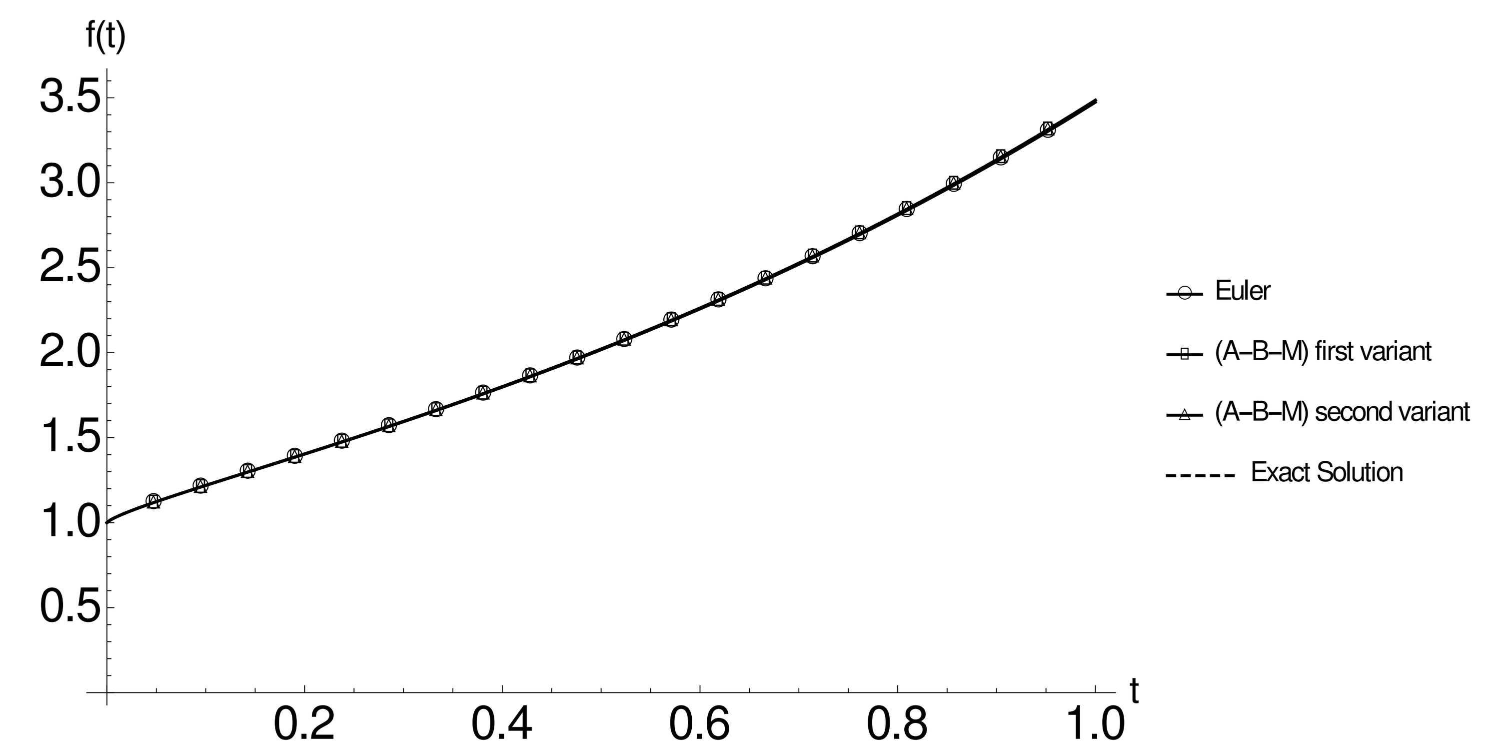
Tablica 5: Error generated by the A-B-M method-second variant.

h	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1/10	7.84e-1	8.92e-2	1.32e-2	1.62e-3
1/50	1.31e-1	8.78e-3	8.55e-4	6.63e-5
1/250	1.81e-2	7.99e-4	5.31e-5	2.66e-6
1/1000	3.21e-3	1e-4	4.8e-6	1.67e-7

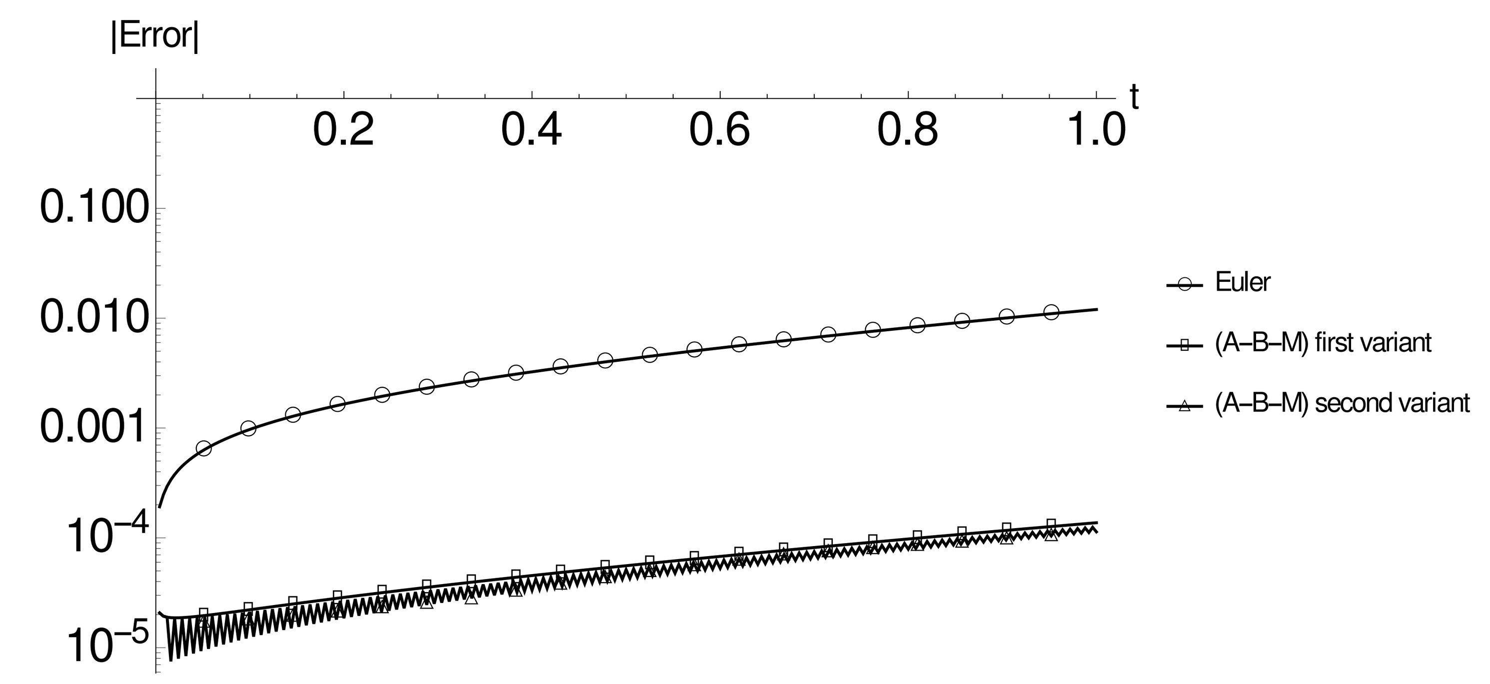
Tablica 6: EOC generated by the A-B-M method-second variant.

h	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$
1/10	1.083	1.463	1.77	1.953
1/50	1.233	1.512	1.771	1.992
1/250	1.253	1.506	1.751	1.998
1/1000	1.253	1.502	1.745	2

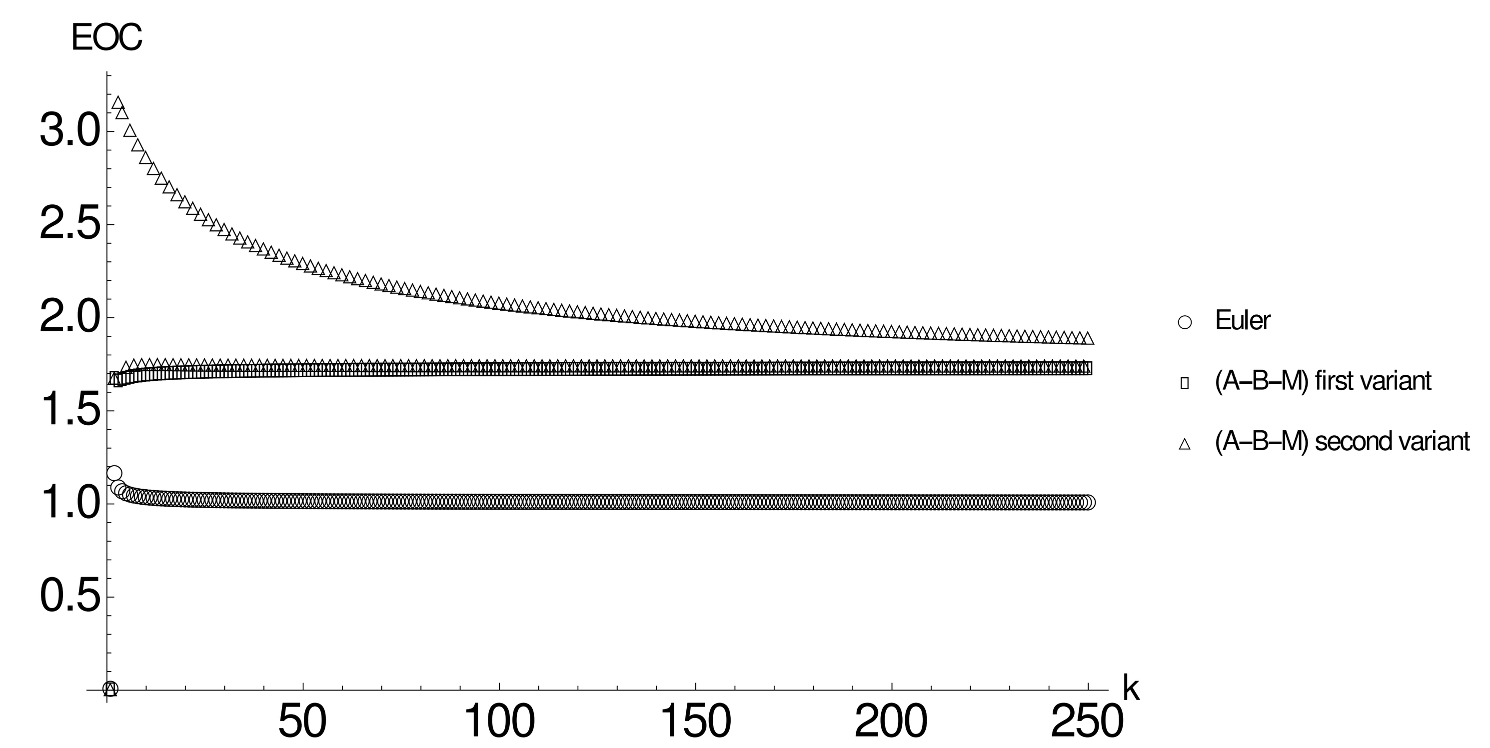
Figure 1 show the numerical and exact solution obtained for value of Caputo derivative $\alpha = 0.75$. In the Figure 2 and 3 we present the graphs of errors and EOC generated by discussed numerical methods.



Rysunek 1: Exact and numerical solutions of ivp (4-5) for $\alpha = 0.75$.



Rysunek 2: The absolute error generated by numerical methods.



Rysunek 3: The absolute EOC generated by numerical methods.

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